From Expander Codes to Higher Cubical Complexes in Quantum Coding Theory

A Short Expository Note with Definitions and Examples

Cassie Ding

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Error-correcting codes are built from a simple *local-to-global* idea: impose many small, local consistency checks, and hope that their collective overlap forces *strong global structure*. In classical coding theory, one of the cleanest realizations of this idea is the Sipser–Spielman expander-code construction [1]: a sparse expanding graph supplies the *geometry*, a short local code supplies the *local rule* at each constraint node, and *expansion* guarantees that local violations cannot hide globally.

Modern quantum LDPC codes ask for a similar *local-to-global principle*, but the geometry has to be richer. In a quantum CSS code there are two families of checks, usually called X- and Z-checks, and they must commute. This *commutation constraint* is much more rigid than in the classical setting: it is no longer enough to place local tests on a graph. One needs a controlled way for different local tests to overlap. This is where *higher-dimensional combinatorial geometry* enters.

The progression in this note is therefore:

$$\text{expander/Tanner code} \longrightarrow \text{left-right Cayley square complex} \longrightarrow \text{cubical complex with local coefficients.}$$

The first step is the familiar *graph-based* picture: vertices and edges organize local constraints. The second step replaces edges by *squares*, so that local neighborhoods look like *tensor-product arrays*; this is the framework behind the first constant-rate, constant-distance, constant-locality locally testable codes (c^3 LTC codes) [4] and the quantum Tanner-code construction [5]. The final step replaces squares by *higher-dimensional cubes* and replaces scalar coefficients by *local coefficient spaces*, or *sheaves*. This gives a flexible chain-complex framework in which the same local-to-global philosophy can be pushed far enough to produce modern qLDPC codes and almost-good quantum locally testable codes. [7]

The point of the note is *not* to survey all technical proofs. Instead, it gives an *intuitive path* through the main objects. The guiding question is:

What geometric structure is needed so that many small local checks force useful global coding properties?

At dimension $t = 1$ the answer is an expanding graph. At dimension $t = 2$ it is a square complex whose vertex neighborhoods carry tensor-code constraints. At general dimension t, it is a higher cubical chain complex with local coefficients. The cells become higher-dimensional, but the real conceptual upgrade is the increasingly organized *pattern of overlaps* among local constraints.

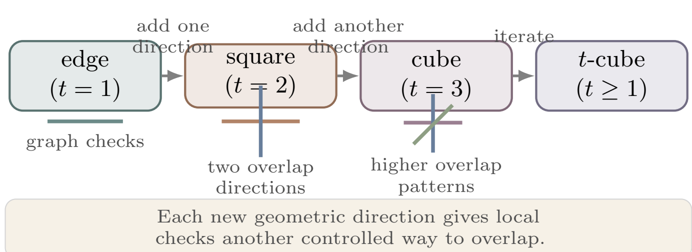

Figure 1: The geometric ladder behind the subject. The muted directions emphasize that the code-theoretic point is not the cell shape itself, but the increasingly rich way local constraints overlap.

1. The one-dimensional model: expander codes.

The classical starting point is a *Tanner code* on a sparse expanding graph. *Expansion* is the mechanism that turns local violations into global detectability.

Definition 1 (Tanner code [2])**.** Let $\Gamma = (L, R, E)$ be a bipartite graph, and assume every right vertex has degree d. Think of L as the set of *variables* and R as the set of *local constraints*. Fix a linear code $C_0 \subseteq \mathbb{F}_q^d$, called the *local code* or *constraint code*. For each $r \in R$, choose an ordering

$$\iota_r : N(r) \xrightarrow{\sim} [d].$$

The associated Tanner code is

$$C(\Gamma, C_0) := \Big\{ x \in \mathbb{F}_q^L : \big(x_v\big)_{v \in N(r)} \circ \iota_r^{-1} \in C_0 \text{ for every } r \in R \Big\}.$$

Thus a word is accepted exactly when every *local view* around a constraint vertex belongs to the same local code C_0.

Definition 2 (Sipser–Spielman expander code [1])**.** Let $\Gamma = (V, E)$ be a finite d-regular graph, usually taken from a family of bounded-degree expanders. Fix a linear *short local code*

$$C_0 \subseteq \mathbb{F}_q^d.$$

Here *short* means that the block length d is the degree of the graph, usually a fixed constant independent of $|V|$; the same constraint code C_0 is reused at every vertex. For each vertex $v \in V$, choose an ordering of the incident edges

$$\iota_v : E(v) \xrightarrow{\sim} [d], \qquad E(v) := \{e \in E : v \in e\}.$$

The Sipser–Spielman expander code associated to (Γ, C_0) is the edge-labeling code

$$C_{\mathrm{SS}}(\Gamma, C_0) := \Big\{ x \in \mathbb{F}_q^E : \big(x_e\big)_{e \in E(v)} \circ \iota_v^{-1} \in C_0 \text{ for every } v \in V \Big\}.$$

Equivalently, each vertex sees the d symbols on its incident edges and checks that this *length-d local word* lies in C_0. The *expansion* of Γ is what prevents many small local errors from remaining globally invisible: a small set of corrupted edge symbols has many boundary vertices, and hence creates many violated local constraints. [1]

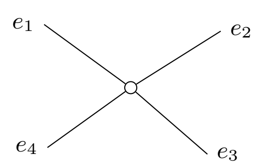

Figure 2: One-dimensional local-to-global propagation. The star around a vertex is the local test domain. Expansion forces a small corrupted set to hit many checks.

2. The first-dimensional lift: left-right Cayley square complexes.

The two-dimensional analogue replaces an *edge* by a *square* and a star by an $A \times B$ array of incident squares. This is the geometry introduced by Dinur–Evra–Livne–Lubotzky–Mozes. [4]

Definition 3 (Left-right Cayley complex)**.** Let G be a finite group and let $A = A^{-1}$ and $B = B^{-1}$ be symmetric generating sets. The left-right Cayley complex

$$\mathrm{Cay}_2(A, G, B)$$

has:

- vertices $X^{(0)} = G$;
- A-edges and B-edges

$$X_A^{(1)} = \big\{\{g, ag\} : g \in G,\ a \in A\big\}, \qquad X_B^{(1)} = \big\{\{g, gb\} : g \in G,\ b \in B\big\};$$

- squares indexed by labeled triples $(a, g, b) \in A \times G \times B$:

$$X^{(2)} = \big\{[a, g, b] : g \in G,\ a \in A,\ b \in B\big\},$$

where the square $[a, g, b]$ is the combinatorial face with vertex set $\{g, ag, gb, agb\}$.

Note that distinct labeled triples may share the same vertex set (e.g., $[a, g, b]$ and $[a^{-1}, ag, b]$ have identical vertices), so the face $[a, g, b]$ must be understood as the labeled cell (a, g, b), not merely its vertex set. The notation is justified by the commuting left/right actions: $a(gb) = (ag)b$. [4]

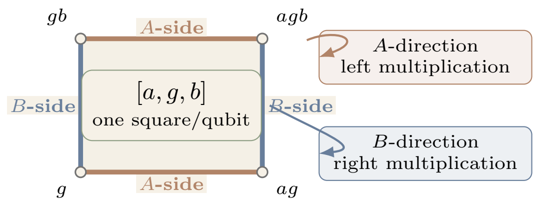

Figure 3: A canonical square in the left-right Cayley complex. Muted clay sides mark the A-direction and muted blue sides mark the B-direction; the controlled overlap of these two directions is the key new ingredient absent from graph-based codes.

The local view around a vertex is naturally a *tensor-code neighborhood.*

Definition 4 (Tensor product code)**.** Let $C_A \subseteq \mathbb{F}_q^A$ and $C_B \subseteq \mathbb{F}_q^B$ be linear codes. Their tensor product is

$$C_A \otimes C_B := \Big\{ M \in \mathbb{F}_q^{A\times B} : M(a,\cdot) \in C_B \ \forall a \in A, \quad M(\cdot, b) \in C_A \ \forall b \in B \Big\}.$$

Equivalently, every *row* lies in C_B and every *column* lies in C_A. (This is a classical construction; the presentation here follows [4].)

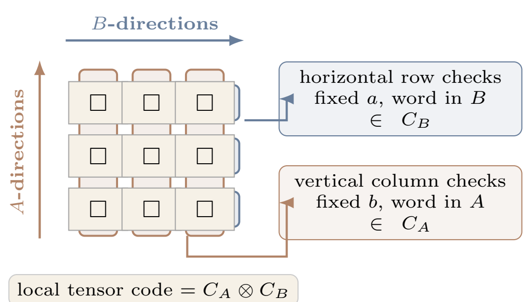

Figure 4: The squares incident to a vertex form an $A \times B$ array. Muted blue horizontal bands show the row checks, while muted clay vertical bands show the column checks. In the convention above, rows are words in C_B and columns are words in C_A.

Definition 5 (Quantum Tanner code)**.** Let $\widetilde{X}$ be the bipartite-covered left-right Cayley complex with vertex set

$$V_{ij} = G \times \{(i, j)\}, \qquad i, j \in \{0, 1\},$$

and with faces inherited from the left-right squares. Fix classical codes $C_A \subseteq \mathbb{F}_2^A$ and $C_B \subseteq \mathbb{F}_2^B$. A quantum Tanner code is the CSS code with

- qubits on the faces of $\widetilde{X}$;
- X-checks at vertices $V_0 := V_{00} \sqcup V_{11}$, where the local check space on the incident faces $Q(v) \cong A \times B$ is $C_A \otimes C_B$;
- Z-checks at vertices $V_1 := V_{01} \sqcup V_{10}$, where the local check space on $Q(v)$ is $C_A^\perp \otimes C_B^\perp$.

The CSS condition $H_X H_Z^T = 0$ is verified by direct face incidence. A face $[a, g, b]$ in the bipartite cover has four vertices: $(g, 00), (ag, 10), (gb, 01), (agb, 11)$. An X-check at $v = (g, 00) \in V_{00}$ is supported on all faces incident to v, namely $Q(v) = \{[a, g, b] : a \in A, b \in B\}$, and defines a vector $u \in C_A \otimes C_B$. A Z-check at $w = (gb_0, 01) \in V_{01}$ is supported on $Q(w) = \{[a, g', b'] : g'b' = gb_0,\ a \in A, b' \in B\}$, and defines $v' \in C_A^\perp \otimes C_B^\perp$. A face $[a, g, b]$ lies in $Q(v) \cap Q(w)$ iff $gb = gb_0$, i.e., $b = b_0$. Thus $Q(v) \cap Q(w) = \{[a, g, b_0] : a \in A\}$, a single column of the $A \times B$ array. On this column, $u(\cdot, b_0) \in C_A$ and $v'(\cdot, b_0) \in C_A^\perp$, so $\langle u(\cdot, b_0), v'(\cdot, b_0)\rangle = 0$. An analogous argument covers the (V_{00}, V_{10}), (V_{11}, V_{01}), and (V_{11}, V_{10}) pairs, in each case reducing the overlap to a single row or column and applying $C_A \perp C_A^\perp$ or $C_B \perp C_B^\perp$. If the underlying degrees are bounded, the family is qLDPC. [5]

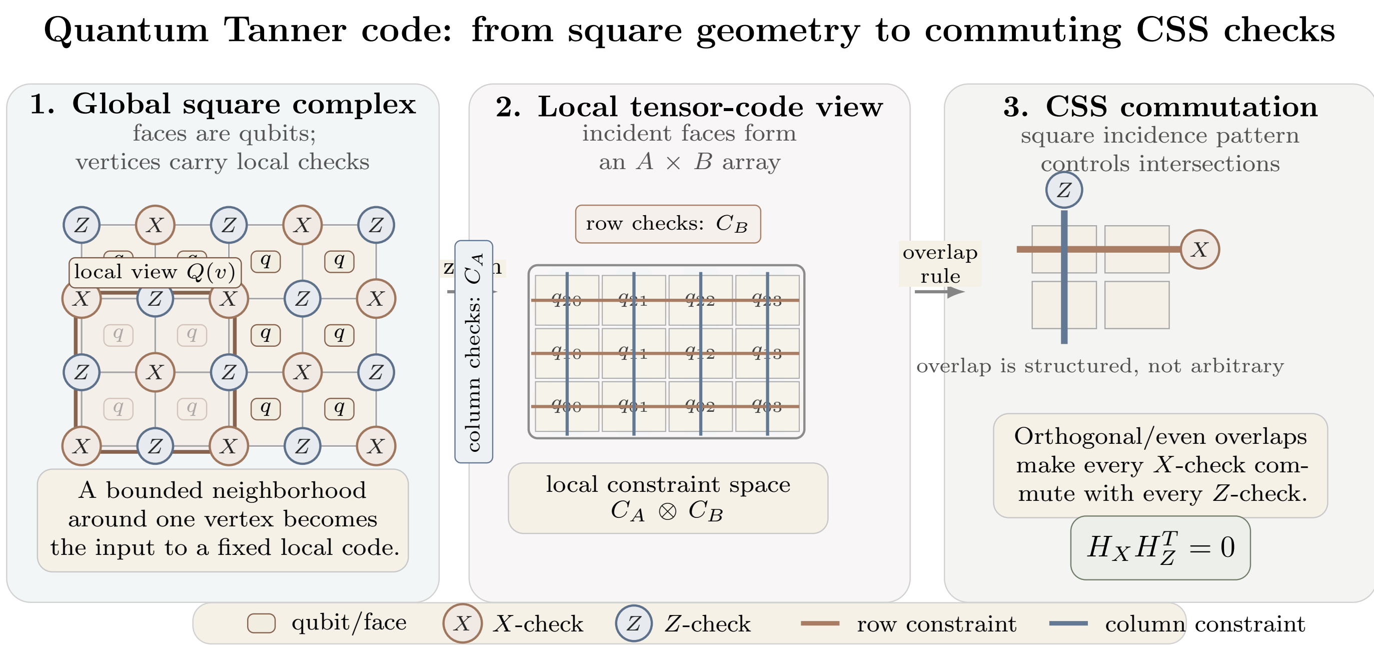

Figure 5: A more visual picture of a quantum Tanner code. The left panel shows the global square complex: qubits sit on faces and the two vertex types support X- and Z-checks. The middle panel zooms into one vertex, where the incident faces form an $A \times B$ tensor neighborhood carrying the local constraint $C_A \otimes C_B$. Here, rows (horizontal, fixing $a \in A$) are checked against C_B, and columns (vertical, fixing $b \in B$) are checked against C_A, consistent with the definition $M(a, \cdot) \in C_B$ and $M(\cdot, b) \in C_A$. The right panel shows the key CSS point: the square-complex incidence pattern makes X-Z overlaps controlled, giving $H_X H_Z^T = 0$. Note that this figure uses the square-complex orientation in which A-edges run horizontally and B-edges run vertically, which is the transpose of the standalone tensor-product figure above.

Definition 6 (qLDPC and qLTC, CSS form)**.** A CSS code with parity-check matrices H_X, H_Z is *qLDPC* if the row and column weights of both matrices are uniformly bounded. It is a *qLTC* with soundness $\rho > 0$ if, for every Pauli error $E = X^{e_X} Z^{e_Z}$ with $e_X, e_Z \in \mathbb{F}_2^n$, the total syndrome weight

$$|\mathrm{synd}(E)| \;:=\; |H_X e_Z| + |H_Z e_X|$$

satisfies $|\mathrm{synd}(E)| \geq \rho \cdot \mathrm{wt}_{\mathrm{corr}}(E)$, where $\mathrm{wt}_{\mathrm{corr}}(E) := \min_{s \in S} \mathrm{wt}(sE)$ is the minimum qubit weight of any Pauli correction of E to the codespace (i.e., the distance of the error coset to the identity). For CSS codes, a *sufficient* condition—employed throughout the chain-complex framework—is that both classical codes $\ker H_X$ and $\ker H_Z$ are individually locally testable with constant soundness; the quantum soundness constant is then bounded below in terms of the two classical constants. [7, 11]

3. Stabilizer codes: two concrete examples.

Before returning to higher cubical complexes, it is useful to isolate three basic notions that recur later: *stabilizers*, *logical operators*, and *logical sectors*. These notions are standard in the stabilizer formalism [9], but they also have a geometric interpretation in CSS codes [10]: in codes arising from chain complexes, stabilizers are local boundaries or coboundaries, logical operators are global cycles not generated locally, and logical sectors are the resulting encoded degrees of freedom. [3]

Intuitively, a Pauli operator is a basic *bit-flip* or *phase-flip* operation on one or more qubits. The operator X flips $|0\rangle$ and $|1\rangle$,

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle,$$

while Z leaves $|0\rangle$ fixed and changes the sign of $|1\rangle$,

$$Z|0\rangle = |0\rangle, \qquad Z|1\rangle = -|1\rangle.$$

Multi-qubit Pauli operators are tensor products of these single-qubit operations. For instance,

$$X_1 Z_3 X_5 = X \otimes I \otimes Z \otimes I \otimes X.$$

The key feature is that Pauli operators either *commute* or *anticommute*. On one qubit,

$$XZ = -ZX,$$

but operators acting on different qubits commute. Thus two multi-qubit Pauli operators commute precisely when the number of qubits on which they anticommute is even. In a stabilizer code, the stabilizers are chosen to commute, so they can be imposed as compatible quantum parity checks.

A stabilizer is therefore a check that every valid code state must pass. Passing the check s means being fixed by it:

$$s|\psi\rangle = |\psi\rangle.$$

Thus the code space is the simultaneous $+1$ eigenspace of all stabilizers: it consists exactly of the states that are fixed by every stabilizer operator.

Let $\mathcal{P}_n$ denote the n-qubit Pauli group, generated by the single-qubit Pauli operators X_i, Z_i together with phases.

Definition 7 (Stabilizers)**.** A *stabilizer group* on n qubits is an abelian subgroup

$$S \subseteq \mathcal{P}_n$$

such that $-I \notin S$. The associated quantum code space is the common $+1$ eigenspace

$$\mathcal{C}(S) := \{\, |\psi\rangle \in (\mathbb{C}^2)^{\otimes n} : s|\psi\rangle = |\psi\rangle \text{ for every } s \in S \,\}.$$

Elements of S are called *stabilizers*. They are the quantum analogue of parity checks: they impose local constraints that every valid code state must satisfy.

For a CSS code with parity-check matrices H_X, H_Z, the stabilizers are generated by the rows of H_X and H_Z: a row of H_X gives an X-type Pauli check, and a row of H_Z gives a Z-type Pauli check. The CSS commutation condition is

$$H_X H_Z^T = 0.$$

Definition 8 (Logical operators)**.** The *normalizer* of the stabilizer group is

$$N(S) := \{\, p \in \mathcal{P}_n : ps = sp \text{ for every } s \in S \,\}.$$

A Pauli operator $p \in N(S)$ preserves the code space because it commutes with every stabilizer. Operators in S act trivially on the code space, so the nontrivial logical Pauli operators are the elements of the quotient

$$\overline{\mathcal{P}} := N(S)/S.$$

Thus a *logical operator* is a Pauli operator that preserves the code space but is not equivalent, on the code space, to a product of stabilizers. Equivalently, logical operators are the physical Pauli operators that act nontrivially on the encoded qubits.

In CSS notation, the X-type and Z-type logical operators are represented by the quotient spaces

$$\ker H_Z \big/ \operatorname{im}(H_X^T), \qquad \ker H_X \big/ \operatorname{im}(H_Z^T).$$

The numerator says "commutes with the opposite checks," while the denominator says "mod out by products of stabilizers."

Definition 9 (Logical sectors)**.** Suppose the code encodes k logical qubits, so that

$$\dim \mathcal{C}(S) = 2^k.$$

Choose representatives of logical Pauli operators

$$\overline{Z}_1, \ldots, \overline{Z}_k, \qquad \overline{X}_1, \ldots, \overline{X}_k,$$

satisfying the usual Pauli commutation relations on the code space. The *logical sectors* with respect to the chosen logical Z-basis are the simultaneous eigenspaces

$$\mathcal{C}_{\mathbf{a}} := \{\, |\psi\rangle \in \mathcal{C}(S) : \overline{Z}_i|\psi\rangle = (-1)^{a_i}|\psi\rangle \text{ for } i = 1, \ldots, k \,\}, \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{F}_2^k.$$

Thus the stabilizers define the code space, while the logical sectors label the different encoded basis states inside that code space.

In many homological or geometric CSS codes, logical sectors can be viewed as *homology classes*: two representatives describe the same logical sector if they differ by a product of local stabilizers.

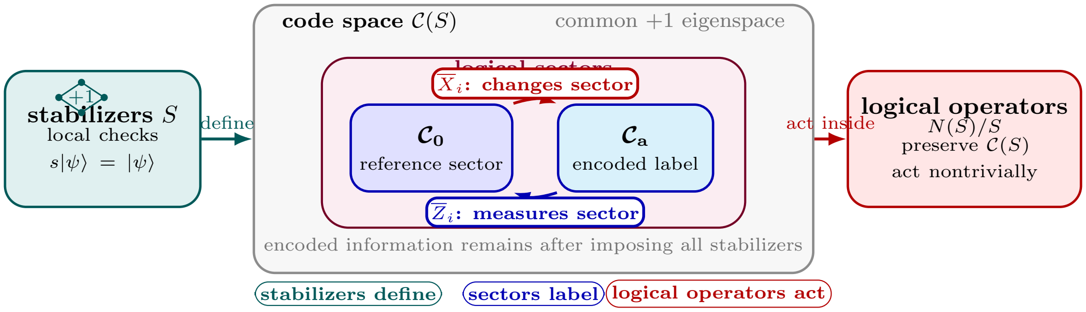

Figure 6: Stabilizers, logical operators, and logical sectors. Stabilizers define the code space $\mathcal{C}(S)$ as a simultaneous $+1$ eigenspace. Logical operators preserve this space but act nontrivially inside it. After choosing a logical basis, the code space decomposes into logical sectors labeled by eigenvalues of the logical $\overline{Z}_i$ operators.

A useful slogan is:

stabilizers define the code space	**logical operators** act inside it	**logical sectors** label encoded information

Example 1 (the 3-qubit repetition code). Take stabilizer generators

$$S_1 = Z_1 Z_2, \qquad S_2 = Z_2 Z_3.$$

The codespace is

$$\mathcal{C} = \mathrm{span}\{|000\rangle, |111\rangle\},$$

so the code encodes one logical qubit, with logical basis

$$|0_L\rangle = |000\rangle, \qquad |1_L\rangle = |111\rangle.$$

A standard choice of logical operators is

$$\overline{X} = X_1 X_2 X_3, \qquad \overline{Z} = Z_1,$$

where any single Z_i serves as an equivalent representative since Z_1, Z_2, Z_3 are all equal modulo S (e.g., $Z_2 = Z_1 \cdot (Z_1 Z_2) \in Z_1 \cdot S$). The two logical sectors for the commuting observable $\overline{Z}$ are therefore the one-dimensional spaces

$$\mathrm{span}\{|0_L\rangle\} \quad \text{and} \quad \mathrm{span}\{|1_L\rangle\}.$$

This is the smallest example where one can see the hierarchy clearly: *stabilizers* define the codespace, *logical operators* act inside it, and *logical sectors* are eigenspaces inside that codespace.

Example 2 (the toric code). Place qubits on the edges of an $L \times L$ square lattice with periodic boundary conditions. [3] For each vertex v and plaquette p, define the star and plaquette stabilizers

$$A_v = \prod_{e \ni v} X_e, \qquad B_p = \prod_{e \in \partial p} Z_e.$$

The global constraints $\prod_v A_v = I$ and $\prod_p B_p = I$ each reduce the rank of the stabilizer generators by one. On the $L \times L$ torus, $|V| = L^2$, $|E| = 2L^2$, $|F| = L^2$, and Euler's formula gives $|V| - |E| + |F| = 0$, so $n := |E| = |V| + |F|$. The stabilizer rank is $(|V| - 1) + (|F| - 1) = |E| - 2$, hence $k = n - \mathrm{rank}(S) = |E| - (|E| - 2) = 2$ encoded logical qubits and four logical sectors. A convenient generating set for the logical operators is given by two noncontractible Z-loops $\overline{Z}_1, \overline{Z}_2$ around the two fundamental directions of the torus, together with the dual noncontractible X-loops $\overline{X}_1, \overline{X}_2$, satisfying

$$\overline{X}_i \, \overline{Z}_j = (-1)^{\delta_{ij}} \, \overline{Z}_j \, \overline{X}_i.$$

Thus the codespace decomposes into four sectors labeled by the eigenvalues of $(\overline{Z}_1, \overline{Z}_2)$:

$$(+,+), \quad (+,-), \quad (-,+), \quad (-,-).$$

Geometrically, *contractible loops* are stabilizers, while *noncontractible loops* are logical operators. This is the topological prototype for the local-to-global viewpoint used throughout the note. Figure 7 packages the toric-code dictionary in the same picture-first style used elsewhere in the note.

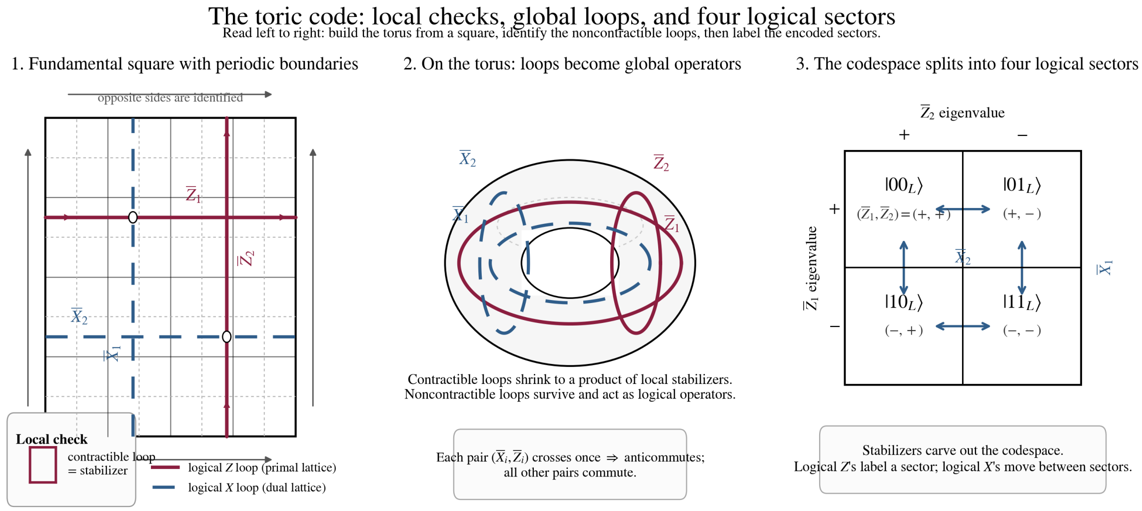

Figure 7: Picture-first toric-code dictionary. Left: the fundamental square with opposite sides identified to form a torus. Middle: the two independent noncontractible directions, supporting logical Z-loops on the primal lattice and dual logical X-loops on the dual lattice. Right: the four logical sectors of the $k = 2$ toric-code codespace, labeled by the eigenvalues of $(\overline{Z}_1, \overline{Z}_2)$.

4. Higher cubical complexes and sheaf coefficients.

Dinur–Lin–Vidick generalize the square-complex picture to arbitrary dimension t. The geometry is encoded as a *graded incidence poset*, and the local algebra is encoded by a *sheaf of coefficient spaces*. [7]

Definition 10 (Higher cubical complex $X(G; \{A_i\})$)**.** Let G be a finite set and let $A_1, \dots, A_t$ be finite sets of permutations of G, each closed under inverse, such that permutations taken from different A_i pairwise commute. The associated cubical complex has vertex set

$$G \times \{0,1\}^t,$$

and, for each subset $S \subseteq [t]$ with $|S| = k$, a k-face of type S is specified by data

$$\big[g; (a_j)_{j\in S}, (b_j)_{j\notin S}\big], \qquad g \in G,\ a_j \in A_j,\ b_j \in \{0,1\},$$

corresponding geometrically to the 2^k vertices

$$\Big\{\big(g \cdot \prod_{j\in T} a_j,\ \beta(T,b)\big) : T \subseteq S\Big\},$$

where $\beta(T,b) \in \{0,1\}^t$ is the binary string whose j-th coordinate is 1 if $j \in T$, 0 if $j \in S \setminus T$, and b_j if $j \notin S$. (The product $\prod_{j\in T} a_j$ is well-defined independent of ordering because elements of different A_i commute by hypothesis.) The complex is naturally 2^t*-partite*; edges in direction i come from A_i. [7]

Definition 11 (Graded incidence poset and local coefficients)**.** A graded poset X with rank function ρ is an *incidence poset* if, whenever $f \preceq f''$ and $\rho(f'') = \rho(f) + 2$, there are an even number of intermediate faces f' with $f \prec\cdot f' \prec\cdot f''$. A system of local coefficients (or sheaf) on X consists of:

- a finite-dimensional vector space V_f for each face $f \in X$;
- linear maps $F_{f\to f'} : V_f \to V_{f'}$ for each $f \succeq f'$ (restriction maps going from higher-rank faces to lower-rank sub-faces), satisfying

$$F_{f'\to f''} \circ F_{f\to f'} = F_{f\to f''} \qquad \text{whenever } f \succeq f' \succeq f''.$$

Terminological note. Because the maps go from larger cells to smaller sub-faces, this is technically a system of local coefficients on the opposite poset—what algebraic topologists would call a *cosheaf*. The term "sheaf" is used throughout the coding-theory literature for this structure; a reader with a sheaf-theoretic background should read "sheaf" as "cellular cosheaf" or "system of local coefficients" throughout. The associated chain groups are

$$C_i(X,F) := \bigoplus_{f\in X(i)} V_f,$$

and the boundary map is

$$\partial_i(u) = \sum_{f' \prec\cdot f} \epsilon(f, f')\, F_{f \to f'}(u), \qquad u \in V_f \subseteq C_i(X, F),$$

where $\epsilon(f, f') \in \{\pm 1\}$ is a sign assignment derived from a chosen orientation of faces (over $\mathbb{F}_2$ the signs are irrelevant and may be omitted; all applications in Sections 4 and 5 work over $\mathbb{F}_2$). The compatibility and incidence conditions, together with the sign assignment, imply the fundamental chain-complex identity $\partial_{i-1}\partial_i = 0$. [7]

Remark (Why the graded incidence poset is needed)**.** This phrase is not just formalism. The *graded* part is what lets us sort faces by dimension and define the chain groups

$$C_i(X, F) = \bigoplus_{f \in X(i)} V_f.$$

Without a grading, there is no coherent meaning to "all i-faces," so the chain complex would not even be indexed correctly. The *incidence* part is what makes "boundary of boundary" vanish: in every rank-2 interval $[f, f'']$, there are exactly two intermediate faces f'_1, f'_2, and the sign assignment is constructed so that

$$\epsilon(f, f'_1)\epsilon(f'_1, f'') + \epsilon(f, f'_2)\epsilon(f'_2, f'') = 0.$$

This cancellation holds over *any* field (it is a signed equality, not just a mod-2 cancellation), which is why the definition adds explicit signs $\epsilon(f, f')$. Over $\mathbb{F}_2$ the signs are redundant; for general $\mathbb{F}_q$ they are essential. Finally, the *poset* language is the right level of generality because cubical faces are organized by inclusion, but the resulting geometry is not simplicial. In the higher cubical complexes $X(G; \{A_i\})$ of Dinur–Lin–Vidick, the face relation is proved to be a transitive graded partial order, and every rank-2 interval has exactly two intermediate faces; that is precisely why the construction yields an incidence poset and hence a genuine chain complex. [7]

Definition 12 (Cycle and cocycle expansion)**.** Let $C_*(X, F)$ be a chain complex with boundary ∂ and coboundary δ. For $k \geq 0$, the *cycle expansion* (equivalently, the local testability parameter for k-chains) is

$$\varepsilon_{\mathrm{cyc}}(k) := \min_{x \in C_k(X,F) \setminus \ker \partial_k} \frac{|\partial_k x|}{\mathrm{dist}(x, \ker \partial_k)},$$

and the *cocycle expansion* is

$$\varepsilon_{\mathrm{cocyc}}(k) := \min_{x \in C^k(X,F) \setminus \ker \delta_k} \frac{|\delta_k x|}{\mathrm{dist}(x, \ker \delta_k)}.$$

Here $|\cdot|$ is the block-Hamming weight determined by the face support. A large $\varepsilon_{\mathrm{cyc}}(k)$ says that any chain far from being a k-cycle has a proportionally large boundary (soundness for k-chains); a large $\varepsilon_{\mathrm{cocyc}}(k)$ gives linear k-distance. These are the quantitative *local-to-global parameters* used in the higher-cubical qLTC analysis. [7]

Definition 13 (Product expansion of a tuple of local codes)**.** Let $C_1, \ldots, C_t \subseteq \mathbb{F}_q^n$ and write

$$C^{(i)} := \mathbb{F}_q^n \otimes \cdots \otimes C_i \otimes \cdots \otimes \mathbb{F}_q^n \subseteq \mathbb{F}_q^{[n]^t}$$

for the subspace of tensors whose i-th slices all lie in C_i. Set

$$C_1 \boxplus \cdots \boxplus C_t \;:=\; \sum_{i=1}^{t} C^{(i)} \;\subseteq\; \mathbb{F}_q^{[n]^t},$$

the (linear) sum of these t subspaces. The tuple $(C_1, \ldots, C_t)$ is *ρ-product-expanding* if every $x \in C_1 \boxplus \cdots \boxplus C_t$ admits a decomposition $x = \sum_{i=1}^t x_i$ with $x_i \in C^{(i)}$ satisfying

$$\sum_{i=1}^{t} |x_i| \;\leq\; \frac{1}{\rho}|x|,$$

where $|\cdot|$ denotes Hamming weight in $\mathbb{F}_q^{[n]^t}$. Intuitively: every element of the sum can be efficiently expressed as a combination of axis-parallel components, with total component weight at most $(1/\rho)$ times the weight of the element. Large ρ means efficient decompositions always exist. The precise formulation in [8] is stated in terms of the number of nonzero axis-parallel lines; the Hamming-weight version above captures the same spirit. This is the higher-arity *local algebraic hypothesis* feeding the cubical constructions. [8]

5. Why the higher-dimensional step matters.

The important point is that higher cubical complexes are *not* just "more boxes." They are a framework in which *geometry and local algebra scale together*. The cubical complex supplies many compatible *overlap patterns*; the sheaf supplies *coefficient spaces and restriction maps*; the expansion parameters then control the resulting chain complex strongly enough to construct qLDPC codes (admitting single-shot decoders [6]) and, at $t = 4$, almost-good qLTCs. [7]

In that sense, the subject develops by repeatedly upgrading the same idea:

$$\text{local consistency on a graph} \Longrightarrow \text{local tensor consistency on squares} \Longrightarrow \text{sheaf-valued consistency on } t\text{-cubes}.$$

That is the conceptual content behind the modern phrase *higher cubical complexes in quantum coding theory*. The cells become higher-dimensional, but more importantly the *local-to-global mechanism* becomes richer, and that richer overlap is exactly what quantum coding theory exploits.

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